Lec 24: 04/20/2010

Jeans Analysis in a Fluid:

We now do a more precise and careful analysis of the evolution of subhorizon perturbations.

First consider a perfect fluid in a static background (means no expansion). Being a continuous medium, the fluid is described by its density $\rho(\vec{r},t)$ and velocity $\vec{v}(\vec{r},t)$ at any point in space and any moment of time (pressure is related to density via the equation of state $P(\vec{r},t)=\omega\rho(\vec{r},t)$).

The relevant equations governing the time-variation of ρ and $\vec{v}$ are Eulerian equations:

$$\begin{cases} \dfrac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0 \\ \dfrac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}+\dfrac{1}{\rho}\vec{\nabla}P+\vec{\nabla}\phi=0 \\ \nabla^2\phi=4\pi G\rho \end{cases}$$

The first equation just describes the continuity, while the second equation is the analogue of Newton's second law describing the net force from pressure and gravity on a tiny element of fluid at point $\vec{r}$ and time t. The third equation is just Poisson's equation for the gravitational field (ϕ being the gravitational potential).

Now lets consider a slightly perturbed homogeneous and isotropic fluid with $\rho = \rho_0 + \rho_1(\vec{r}, t)$, $\phi = \phi_0 + \phi_1$, and $\vec{v} = \vec{v}_0 + \vec{v}_1(\vec{r}, t)$. ρ_0 and $\vec{v}_0$ are constants, and ρ_1, $\vec{v}_1$ represent small perturbations. Equation of state implies that $P = P_0 + P(\vec{r}, t)$.

We the find the following equations for the perturbations:

$$
\begin{cases}
\dfrac{\partial \rho_1}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{v}_1 = 0 \\
\dfrac{\partial \vec{v}_1}{\partial t} + \dfrac{v_s^2}{\rho_0} \vec{\nabla} \rho_1 + \vec{\nabla} \phi_1 = 0 \\
\nabla^2 \phi_1 = 4\pi G \rho_1
\end{cases}
$$

Here $v_s = \omega^{1/2}$ is the speed of sound in the fluid.

Taking the partial derivative with respect to time of the first equation, and using the other two equations, we find a wave equation for ρ_1:

$$\frac{\partial^2 \rho_1}{\partial t^2} - v_s^2 \nabla^2 \rho_1 = 4\pi G_\circ \rho_1$$

After Fourier decomposition of ρ_1, for a given mode:

$$\rho_{1,k} = \delta_k \rho_\circ = A \exp(-i \vec{k} \cdot \vec{r} + i\omega t)\, \rho_\circ$$

We find:

$$\boxed{\ddot{\delta}_k + (v_s^2 k^2 - 4\pi G \rho_\circ)\, \delta_k = 0} \qquad \left(\dot{} \equiv \frac{d}{dt} \right)$$

This gives rise to the following dispersion relation:

$$\boxed{\omega^2 = v_s^2 (k^2 - k_J^2) \qquad k_J \equiv \left(\frac{4\pi G \rho_\circ}{v_s^2} \right)^{\frac{1}{2}}}$$

Where k_J is called the "Jeans wavenumber" and $\lambda_J = \frac{2\pi}{k_J}$ is the "Jeans wavelength".

Note that for $k > k_J$ $(\lambda < \lambda_J)$ we have $\omega^2 > 0$, hence oscillatory solutions for δ_k. On the other hand, for $k < k_J$

$(\lambda > \lambda_J)$, we have $\omega^2 < 0$, thus exponentially growing and decaying solutions.

This analysis can be extended to an expanding universe. In this case:

$$\rho_{0(t)} = \rho_0(t_0)\left[\frac{a(t)}{a(t_0)}\right]^3 , \quad \vec{v}_0 = \frac{\dot{a}(t)}{a(t)}\vec{r} , \quad \vec{\nabla}\phi_0 = \frac{4\pi G_0}{3}\vec{r}$$

We then find:

$$\boxed{\ddot{\delta}_k + 2H_{(t)}\dot{\delta}_k + \left[\frac{v_s^2 k^2}{a_{(t)}^2} - 4\pi G \rho_{0(t)}\right]\delta_k = 0}$$

The physical wavenumber is $\frac{k}{a}$, and the second term on the left-hand side the equation represents the damping, both as a result of expansion.

So far, we have considered a single component fluid. In general, we have a multi-component fluid (baryons, photons, dark matter, etc). In this case:

$$\rho_0 = \sum_i \rho_{0,i} \quad , \quad \delta_k^i = \frac{\delta\rho_{0,i}}{\rho_0}$$

The fraction of each component is given by $\varepsilon_i \equiv \frac{\rho_{0,i}}{\rho_0}$. As for perturbations, we are interested in δ_k^i's for components that are non-relativistic species (relativistic particles like photons are not attracted toward overdense regions since they move with the same speed c).

For a given component "i", we have:

$$\boxed{\ddot{\delta}_k^i + 2H_{(t)}\dot{\delta}_k^i + \left[\frac{v_{s,i}^2 k^2}{a_{(t)}^2} - 4\pi G \rho_{0(t)} \sum_j \varepsilon_j \delta_k^j\right] = 0}$$

Here $v_{s,i}$ is the sound speed of the i-th components and the sum is over all non-relativistic components.

For dark matter (a typical weakly interacting particle) decoupling happens at $t \sim 1$ sec. After that we have $v_s = 0$.

On ther hand, baryons remain coupled to the plasma by the pressure provided by photons until the recombination $t_{rec} \sim 400{,}000$. Therefore $v_s = \sqrt{\frac{1}{3}}$ for baryons before recombination.

Now lets consider perturbations in dark matter in a radiation-dominated universe. ρ_0 is dominated by radiations, which implies that $\varepsilon_{DM} \ll 1$. As a result, we find the following equation for δ_{DM} (note that $v_s = 0$ after 1 second):

$$\ddot{\delta}_{DM} + \frac{1}{t}\dot{\delta}_{DM} = 0 \qquad \left(H = \frac{1}{2t} \text{ in a radiation dominated universe}\right)$$

The solution is;

$$\delta_{DM}(t) = \delta_{DM}(t_i)\boxed{1 + c\ln\left(\frac{t}{t_i}\right)}$$

Here t_i is an initial time, and "c" is a constant. Thus in a radiation dominated universe dark matter perturbations can grow logarithmically, and even then, if they have a non-zero velocity initially.

This is quite important. Dark matter does not feel any pressure, and hence its perturbations can grow due to gravitational attraction. But this is not significant until the universe becomes matter dominated. Recall that this happens at $t_{EQ} \sim 100{,}000$ yr.

The situation is worse for baryons because they remain coupled to the plasma, hence pressure gradient opposes gravitational attraction, until $t_{rec} \sim 400{,}000$ yr. After recombination, however, perturbations in baryons can grow quickly.